1. Introduction

In the 21st century computers are in the process of changing all aspects of our lives. That includes also mathematics - both the way we do it and even the kind of problems that interest us. The ability to make computations far beyond anything that was possible without the aid of computers has opened up previously inaccessible areas of research to anyone equipped with a computer,   suitable software and some mathematical knowledge. This has lead to the creation of sophisticated programs intended to for dealing with all aspects of computational mathematics - symbolic manipulation, numerical computation and visualization (including interactive one). A leading program of this type is *Mathematica* produced by Wolfram Research. *Mathematica* is, of course, not only a tool of research and exploring new areas of computational mathematics but also an extremely effective aid in studying traditional ones. The aim of this essay is to give a brief introduction to some of the possibilities offered by this remarkable program.

Basic information about Mathematica

The main web-page

The main website for current information, useful projects, plug-ins, learning center
 and documentation is www.wolfram.com . At this website one can also find a free CDF-Player, thousands of demonstrations with source files and educational videos.

Mathematica is one of the most powerful and sophisticated systems for symbolic and numerical computation and visualization. But it is also more than that. Wolfram Research used to describe *Mathematica* as a "system for doing mathematics by computer" but it has since changed this to " the only development platform fully integrating computation into complete work-flows". In other words, Wolfram Research now conceives *Mathematica* as essentially a universal tool for almost every purpose - symbolic and numerical computation and programming being a central aspect but not the whole story. For example, *Mathematica* is also an advanced technical typesetting tool, which can produce mathematical documents of quality comparable to T_EX and Latex but in a fully WISIWIG way. At the same time, these documents can contain "live" mathematical formulas and graphics that can be send to another person (a collaborator, a student or a professor) who can use them to verify the correctness of results or to perform additional computations etc.

In terms of its overall abilities *Mathematica* currently has no comparable rival (except, perhaps, for suites of several applications). But if we restrict our attention only to computation, there are other programs that can do similar things. As a mathematical tool, *Mathematica* is a "general purpose" system. It is very strong in both symbolic computation and also in numerical one. It has very many powerful specialized functions for subjects as different as polynomial algebra, graph theory, statistics of financial mathematics. However, in some areas (particularly in pure mathematics) there are specialized programs (such as MAGMA, Singular, Macaulay2 etc.) which can do some things that *Mathematica* cannot do without additional programming or can do them faster (which in certain situations can be crucial).

The official distributor of Mathematica in Poland is the company Gambit
<http://www.mathematica.pl>.

Wolfram Research sites and projects

Wolfram Research has a number of sites and projects of interest both to *Mathematica* users and the general public:

Mathworld <http://mathworld.wolfram.com/>

Mathworld is a very useful and extensive web resource with definitions, examples and main theorems in mathematics (and often - *Mathematica* files in the NB format, i.e., Mathematica notebooks .)

Wolfram Alpha <http://www.wolframalpha.com/>

Wolfram Alpha - *Mathematica* (and more) for everyone!

Wolfram Demonstrations Project <http://demonstrations.wolfram.com>

Education

Wolfram Research has an extensive educational program. To register for on-line Wolfram Educational Group (WEG) seminars and classes, you visit
  <http://www.wolfram.com/services/education/calendar.cgi>
The Wolfram Education Group (WEG) offers a wide range of free on-line seminars featuring the latest version of *Mathematica* (see http://www.wolfram.com/services/education/seminars/).

There are conferences/master classes/other events in Europe and in particular, in Poland. In Warsaw, ICM (the interdisciplinary center at UW) has Mathematica and provides some training.

The Mathematica Journal

 The Mathematica Journal <http://www.mathematica-journal.com/> is an on-line journal with research and educational papers.

Other *Mathematica* related sites

There is an excellent and free introduction to *Mathematica* programming due to Leonid Shifrin:
http://www.mathprogramming-intro.org/
Also the *Mathematica* Guidebooks <http://www.mathematicaguidebooks.org/index.shtml> provide a wealth of applications and examples, particularly from physics but also from other areas. Unfortunately the guidebooks are still not fully compatible with *Mathematica* 6 and later versions so they are only suitable for users who are advanced enough to update the relevant parts themselves.

Finally there is an excellent *Mathematica* discussion group (known as the *MathGroup*)
http://groups.google.com/group/comp.soft-sys.math.mathematica/topics?pli=1
You can ask any questions about any aspect of *Mathematica* and get a variety of answers from *Mathematica* experts, including Wolfram Research employees. Sometimes you can even have your math problems solved for you.
This is really the best *Mathematica* centered resource on the Web, if you know how to make use of it.

SELECTED EXAMPLES OF WHAT MATHEMATICA

CAN DO

Computational applications

Built-in functions

A large number of problems (even some "real life" ones) can be solved by simply applying one of Mathematica's built-in "functions" (the word "function" in connection with *Mathematica* is used in a somewhat different sense than in mathematics, closer to what in other programming languages is called a "procedure"). Among the most useful of these functions are Solve and Reduce. These are very general functions that uses a large ensemble of advanced algorithms to solve all kinds of equations and inequalities, many of which would appear unsolvable even to people with good knowledge of mathematics. We will looks at some remarkable examples of the sort of thing that can be accomplished with Reduce.
(In version 8 the functions Solve became very enhanced and can solve many of the problems that Reduce can, but it often uses somewhat different techniques. Probably the main difference between the two functions is that Reduce always attempts to return the complete solutions of a problem while Solve will in some situations return a partial solution.)

Perhaps the most basic and frequently performed task in mathematics is solving equations. Reduce can be used, of course, to solve equations (and systems of equations) far too complicated to solve by hand. We start by looking at a simple cubic equation, which *Mathematica* can solve using the famous formula of del Ferro (often attributed to Tartaglia):

```
Reduce[x^3 - x + 1 == 0, x, Cubics → True]
```

$$x = -\sqrt[3]{\frac{2}{3\left(9-\sqrt{69}\right)}} - \frac{\sqrt[3]{\frac{1}{2}\left(9-\sqrt{69}\right)}}{3^{2/3}} \quad \bigvee$$

$$x = \left(\left(1+i\sqrt{3}\right)\sqrt[3]{\frac{1}{2}\left(9-\sqrt{69}\right)}\right)\Big/\left(2\times3^{2/3}\right) + \frac{1-i\sqrt{3}}{2^{2/3}\sqrt[3]{3\left(9-\sqrt{69}\right)}} \quad \bigvee$$

$$x = \left(\left(1-i\sqrt{3}\right)\sqrt[3]{\frac{1}{2}\left(9-\sqrt{69}\right)}\right)\Big/\left(2\times3^{2/3}\right) + \frac{1+i\sqrt{3}}{2^{2/3}\sqrt[3]{3\left(9-\sqrt{69}\right)}}$$

We can also ask *Mathematica* to compute the real root only:

```
Reduce[x^3 - x + 1 == 0, x, Reals, Cubics → True]
```

$$x = -\sqrt[3]{\frac{2}{3\left(9 - \sqrt{69}\right)}} - \frac{\sqrt[3]{\frac{1}{2}\left(9 - \sqrt{69}\right)}}{3^{2/3}}$$

We can compute it's numerical value to arbitrary precision:

```
N[%, 30]
```

$x = -1.32471795724474602596090885448$

From the work of Abel and Galois it is known that no solutions of the above kind (in terms of radicals) can be given for polynomial equations of degree higher than 5. However, this does not stop *Mathematica* :

```
Reduce[x^5 - x + 1 == 0, x]
```

$$x = \text{Root}\left[\#1^5 - \#1 + 1 \&, 1\right] \bigvee x = \text{Root}\left[\#1^5 - \#1 + 1 \&, 2\right] \bigvee$$
$$x = \text{Root}\left[\#1^5 - \#1 + 1 \&, 3\right] \bigvee x = \text{Root}\left[\#1^5 - \#1 + 1 \&, 4\right] \bigvee x = \text{Root}\left[\#1^5 - \#1 + 1 \&, 5\right]$$

Again the values can be computed to arbitrary precision:

```
N[%, 30]
```

$x = -1.16730397826141868425604589985 \bigvee$
$\quad x = -0.181232444469875383901800237781 - 1.08395410131771066843034449298 1\, i \bigvee$
$\quad x = -0.181232444469875383901800237781 + 1.08395410131771066843034449298 1\, i \bigvee$
$\quad x = 0.764884433600584726029823187709 - 0.352471546031726249317947091403\, i \bigvee$
$\quad x = 0.764884433600584726029823187709 + 0.352471546031726249317947091403\, i$

Reduce can also deal with purely symbolic problems. For example, consider the quadratic equation $a\,x^2 + b\,x + c = 0$. Let's obtain the well known condition for it to have two equal roots.

$$\text{Reduce}\left[\exists_{x, a\,x^2+b\,x+c=0} \bigvee_{y, a\,y^2+b\,y+c=0} x = y, \{a, b, c\}\right]$$

$$(a = 0 \wedge b \neq 0) \bigvee \left(a \neq 0 \wedge c = \frac{b^2}{4\,a}\right)$$

Let's now try something non-polynomial. Here is a trigonometric equation which we ask Reduce to solve for a range of values of the variable where there is a finite number of solutions:

```
Reduce[Cos[x] == Sin[x] && 0 < x < 2 Pi, x] // FullSimplify
```

$4\,x = \pi \bigvee 4\,x = 5\,\pi$

We can also obtain a complete solution:

```
Reduce[Cos[x] == Sin[x], x] // FullSimplify
```

$c_1 \in \mathbb{Z} \wedge (\pi\,(8\,c_1 - 3) = 4\,x \bigvee 8\,\pi\,c_1 + \pi = 4\,x)$

Here is an equation that seems impossible to solve by hand, but Reduce can do it:

```
Reduce[Cos[Cos[x]] == Sin[Sin[x]] && Abs[x] < 1, x]
```

$x = \text{Root}[\{\cos(\cos(\#1)) - \sin(\sin(\#1)) \&, 0.7853981633974483096156608458198757210492923498437764552437 - 0.4663385348278305845718632848784660354269560408360176474839 \, i\}] \; \vee$
$\quad x = \text{Root}[\{\cos(\cos(\#1)) - \sin(\sin(\#1)) \&, 0.7853981633974483096156608458199 + 0.4663385348278305845718632848785 \, i\}]]$

```
N[%, 10]
```

$x = 0.7853981634 - 0.4663385348 \, i \; \vee \; x = 0.7853981634 + 0.4663385348 \, i$

All the solutions are complex numbers. Reduce can prove that there are no real solutions:

```
Reduce[Cos[Cos[x]] == Sin[Sin[x]], x, Reals]
```

False

Here is a completely different kind of equation, this time over the integers.

```
Reduce[n! + n == 726 && n > 0, n, Integers]
```

$n = 6$

Let's now try something harder - a "real life" problem. There is an Internet forum, called the *Math-Group*, where people post questions about *Mathematica*, including mathematical questions they try to solve with *Mathematica.* Solutions are posted by other users including some of the staff of Wolfram Research. One such question was posted by Ivan Smirnov in January 2011 (the whole thread can be found here: Smirnov's problem

Are there any integer solutions of $x^{10} + y^{10} + z^{10} = t^2$?

It is easy to find trivial solutions where two of the three variables x, y, z are zero, so let's look for such solutions. Reduce cannot solve the complete problem but it can quickly verify that there are no solutions for $t \le 10^4$.

```
Reduce[x^10 + y^10 + z^10 == t^2 && 0 ≤ x && 0 < y && x ≤ y && y ≤ z && 1 ≤ t ≤ 10^4,
  {x, y, z, t}, Integers] // Timing
```
{1.53308, False}

In fact (after changing certain settings which limit the number of cases Reduce will consider, one can verify that there are no solutions for $1 \le t \le 10^{10}$.

Reduce can be very useful in many undergraduate courses. For example, in Analysis 1 one often needs to prove that a certain integer sequence is monotonic. Consider, for example, the problem

```
Reduce[n^{1/n} > (n + 1)^{\frac{1}{n+1}} && n > 0, n, Integers]
```

$n \in \mathbb{Z} \bigwedge n \geq 3$

Programming

Although many problems can be solved just by applying built-in functions, in many cases there is no

built in function that will do all the work itself. In such cases we need to do our own programming. Here are some recent problems taken from the *MathGroup.*

Find three 2 - digit prime numbers such that :
(i) The average of any two of the three is a prime number, and
 (ii) The average of all three is also a prime number

There is no *Mathematica* function that will automatically answer a question like this, but the answer can be found with a few lines of *Mathematica* code:

```
ls1 = Select[Range[11, 99], PrimeQ];
ls2 = Tuples[ls1, {3}];
ls3 = DeleteCases[Union[Sort /@ ls2], {___, x_, ___, x_, ___}];
Select[ls3, And @@ PrimeQ /@ Mean /@ Partition[#1, 2, 1, {1, 1}] &]
```

$$\begin{pmatrix} 11 & 23 & 71 \\ 11 & 23 & 83 \\ 11 & 47 & 71 \\ 13 & 61 & 73 \\ 17 & 29 & 89 \\ 23 & 59 & 83 \\ 29 & 53 & 89 \end{pmatrix}$$

Here is another question from the *MathGroup*:

What' s the easiest way to determine the length of the repeating cycle for decimal expansions of fractions? For example, 1/7 == 0.14285714285714285714. . .so the length of its repeating cycle (142857) is 6. For 1/3 the length of the cycle is obviously 1. For some fractions, e.g., 1/4, the decimal expansion is not cyclical (in base 10).

We use programming to define a function lengthOfCycle. We make use of several built-in *Mathematica* functions, in particular, IntegerExponent and MultiplicativeOrder. If we could not use these functions the program would have to be much longer, more complicated and less efficient.

```
lengthOfCycle[x_Rational] :=
 Module[{n = Denominator[x], a, b}, a = IntegerExponent[n, 2];
  b = IntegerExponent[n, 5];
  MultiplicativeOrder[10, n / (2^a * 5^b)]]
```

```
lengthOfCycle[1 / 7]
```

6

In certain situations programs written in *Mathematica* programming language can be much slower than programs written in typed and compiled languages such as C, Java, etc (although, of course, writing such programs in *Mathematica* is almost always much quicker). However, for many types of programs this difference can be greatly reduced by "compiling". Not every kind of *Mathematica* program can be successfully compiled but when it can, this can make a very big difference to performance. Here is an example where a non-compiled Mathematica program performs rather poorly. We will give only a compiled version, that is very fast. The program constructs an Ulam spiral and is due to Daniel Lichtblau of Wolfram Research:

```
ulamSpiral = Compile[{{len, _Integer}}, Module[
    {dat = Range[len], x = 0, y = 0, shift = 1, i = 0, j = 0, xincr = {1, 0, -1, 0},
     yincr = {0, 1, 0, -1}, shiftincr = {0, 1, 0, 1}}, Select[Table[i++;
     If[i ≥ shift, i = 1; shift += shiftincr[[Mod[j, 4, 1]]]; j++;];
     x += xincr[[Mod[j, 4, 1]]]; y += yincr[[Mod[j, 4, 1]]];
     If[PrimeQ[dat[[num]]], {x, y}, {1000, 1000}], {num, len}],
    Abs[First[#]] < 1000 &]], {{PrimeQ[_], True | False}}];
```

Making a spiral with 60 000 points takes only a fraction of a second:

```
ls = ulamSpiral[60 000]; // Timing
```

{0.088024, Null}

We can view the spiral as a collection of points:

```
ListPlot[ls]
```
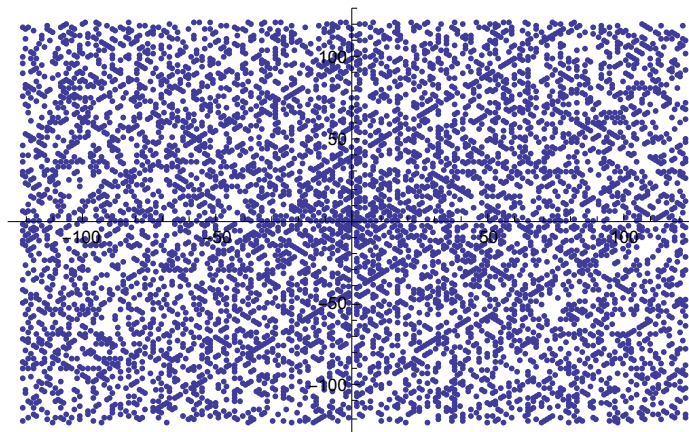

or of lines

```
ListLinePlot[ls]
```
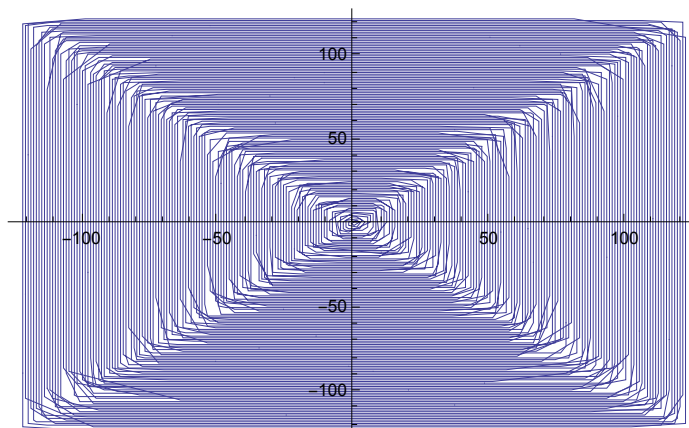

In *Mathematica* version 8, this can be speeded up further by using the option
CompilationTarget → "C", which however, requires that a C-compiler be installed on the computer.

Specialized Mathematics

Mathematica contains a large number of specialized functions for various areas of mathematics, ranging from Group Theory and Number Theory to Statistics (a vast number of statistical distributions are available as built-in functions) and financial mathematics. A particularly interesting aspect is Mathematica's ability to obtain live financial data from the Internet and analyze them by mean of a variety of specific financial functions. As an example we compare the performance of the value of the index of the American NASDAQ stock exchange (on which most US technology companies are represented) and of Apple Computer Inc.

```
data1 = FinancialData["^IXIC", "Jan. 1, 2007"];

g1 = DateListPlot[data1, Joined -> True, PlotLabel → "NASDAQ Composite"];

data2 = FinancialData["AAPL", "Jan. 1, 2007"];

g2 = DateListPlot[data2, Joined -> True, PlotLabel → "Apple"];

GraphicsGrid[{{g1, g2}}]
```
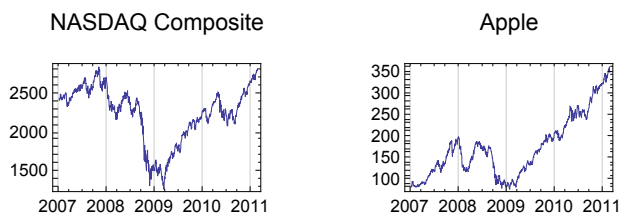

Graphics

Mathematica has remarkable graphic capabilities. Here are some mathematical graphics (due to the author of this note). The first one is related to the subject of iteration of functions in the complex plane and "fractals", the second shows the so called "Witten-Smale" flows on the tilted torus. To create this graphic one needs to solve differential equations on the torus.
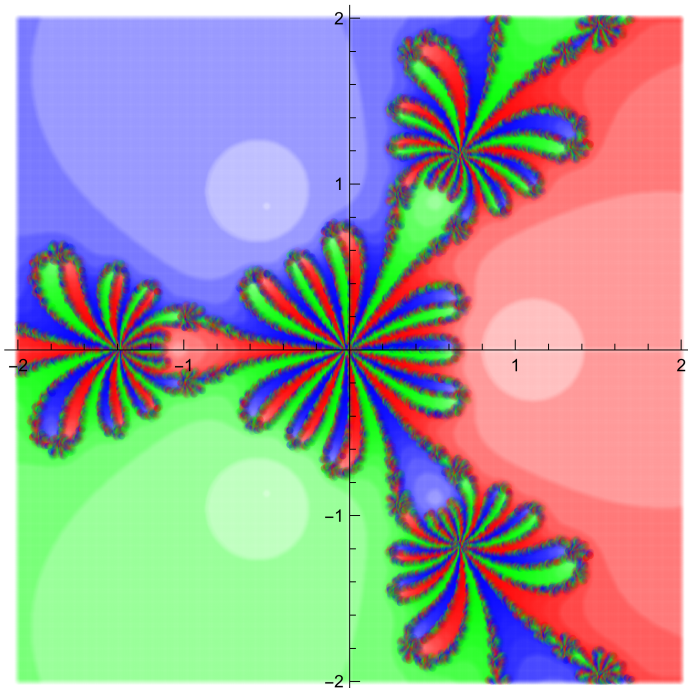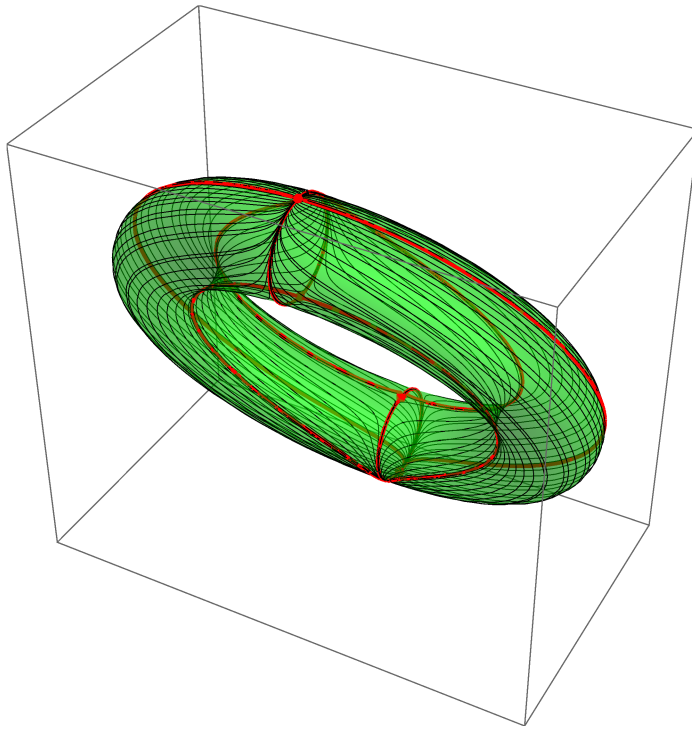
Beyond Graphics - Interactive Dynamics

A completely new set of features, that have nothing quite similar in other programs, appeared in *Mathematica* 6. These involve interactive "dynamic" abilities that are difficult to describe in a static format, but can be seen on the page below.
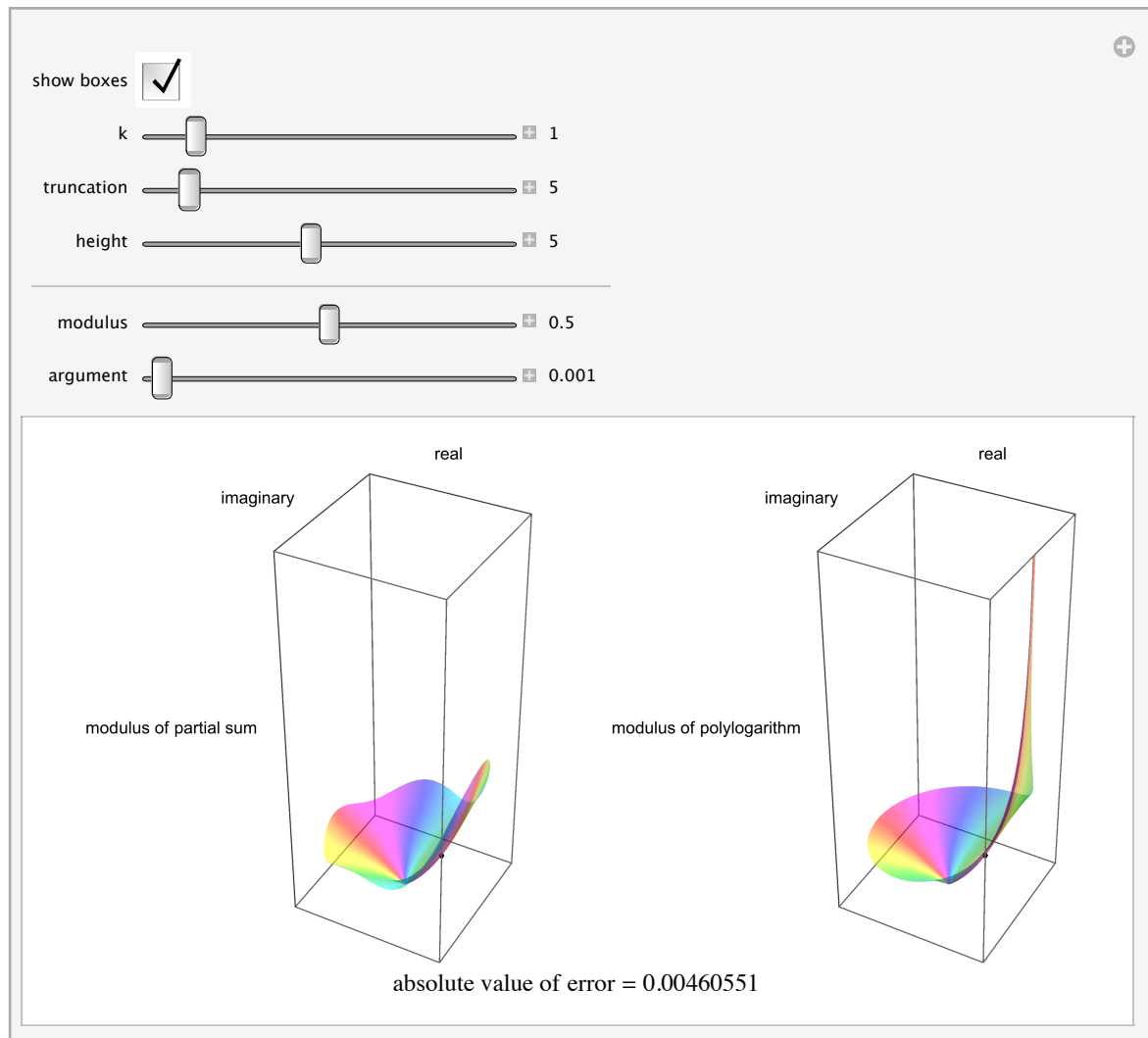
Wolfram Demonstrations Project

It is difficult to describe on a static page how remarkably useful this functionality is in all kinds of situations, including both research and teaching. A single interactive *Mathematica* notebook can replace dozens static pictures. I will mention here just two examples, from two different fields.

The first is Mathematical finance, more precisely, the pricing on financial options. The famous classical model of Black-Scholes which gives a formula for the value of a European call or put option, is known to be based on unrealistic assumptions about how markets work and can be badly wrong in certain circumstances. There are many other models, with more realistic assumptions. These models generally involve many parameters, and the price of an option depends on their values. For certain values of the parameters the price is going to be very close to the Black-Scholes price but for others it may be quite far from it. This problem is very difficult to study in the general setting. What is normally done, is that one chooses certain values of parameters and compares the difference between the prices obtained from both models. For every choice of parameters one needs to make a new comparison, which is usually done graphically. That means that a large number of different settings and corresponding graphics are needed. Using *Mathematica*, on the other hand, a single interactive "dynamic" object will suffice. Below we see two graphics comparing the prices of an option based on the Black-Scholes model (green) and the Kou jump diffusion model (Brown). As we can see, for certain parameter values they prices are very close but for others they can differ by a great deal.
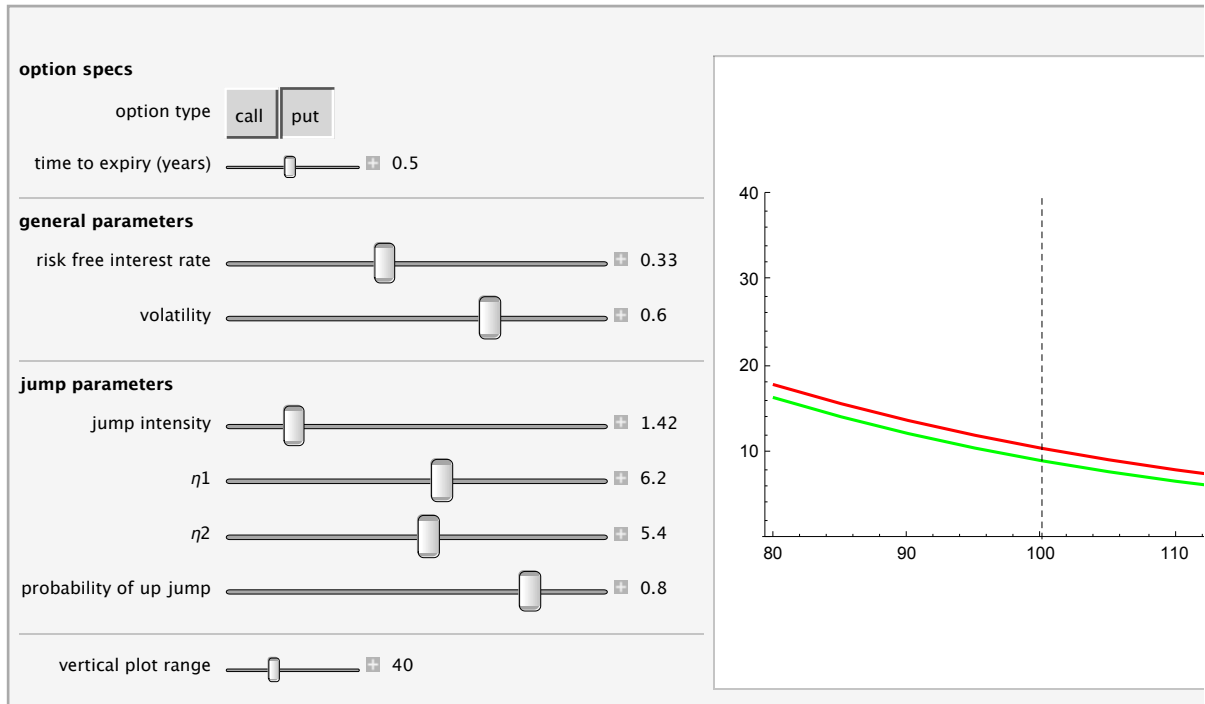
Below we see another example. This time we study the convergence of the power series power series $\sum_{n=1}^{\infty} \frac{z^n}{n^k}$ on the unit disk $|z| \leq 1$ for $k = 0, 1, 2 \ldots$. Here we only see the case $n = 1$, when the series converges in the unit disk and everywhere on the boundary except at the point $z = 1$. The graphic on the left shows the complex values over the unit disk of a finite sum of terms of the series, on the right we see the values of the analytic function defined by the series. The modulus of the complex values of a function is represented by the height of the graph and the argument by colour. In the first graphic a shorter finite sum is used, in the second a longer. We can see the improvement in the approximation and the singular behaviour at $z = 1$. We can also interactively choose any point on the unit disk and see the modulus of the difference between the values of the corresponding finite sum and the infinite series. Again, this is the sort of thing that would be very difficult to reproduce by other means.

show boxes ✓

k ▭ 1

truncation ▭ 5

height ▭ 5

modulus ▭ 0.5

argument ▭ 0.001

real

imaginary

real

imaginary

modulus of partial sum

modulus of polylogarithm

absolute value of error = 0.00460551

The *CDF Format and the CDF Player*

With version 8 of *Mathematica* a new file format for *Mathematica* files was introduced.

The CDF player makes it easy to use *Mathematica* in class or at home even when ones students do not have *Mathematica* themselves. It is a free program that can be downloaded from Wolfram's web site.
Wolfram CDF Player

Wolfram Alpha

`Wolfram Alpha` - *Mathematica* for everyone!

Wolfram Alpha received a great deal of publicity when it first appeared and it may even be better known than *Mathematica,* but not many people seem to realize that Wolfram Alpha is based on *Mathematica* and, in effect, provides the general public with free access to much of Mathematica's functionality. Moreover, it does not require learning the *Mathematica* syntax. Indeed, the most famous aspect of Wolfram Alpha is its ability to use "free form" mathematical input - just type in what you want Wolfram Alpha to do for you in ordinary English (other language input is planned for the future) and Wolfram Alpha will attempt to guess what you want and then will use *Mathematica* to obtain the answer. In fact, the answers returned my Wolfram Alpha are generally more complete then *Mathematica* would normally return (they can all be obtained with *Mathematica* but it may require several commands or even some programming). Here is an example of computing the integral $\int x \log(x)\, dx$ with Wolfram Alpha. I just typed: "indefinite integral of x log(x) " and received the output shown below. In versions of *Mathematica* earlier than 8, a very precise syntax would be needed to obtain the same result (version 8 of *Mathematica* can also use "free form" input).

WolframAlpha computational... knowledge engine

indefinite integral of x log(x)

Assuming "log" is the natural logarithm | Use the base 10 logarithm instead
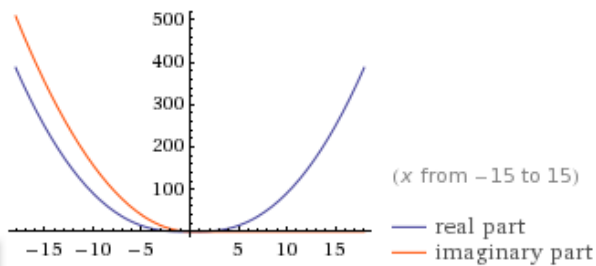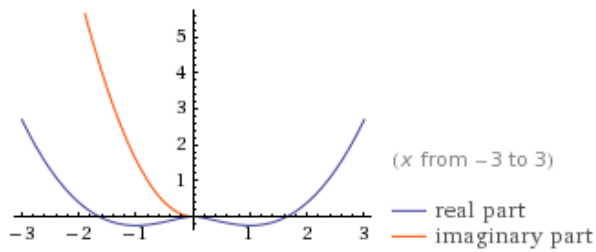
Indefinite integral: [Show steps]

$$\int x \log(x)\, dx = \frac{1}{4}\, x^2\, (2 \log(x) - 1) + \text{constant}$$

$\log(x)$ is the natural logarithm »

Plots of the integral:

(x from −3 to 3)

— real part
— imaginary part

(x from −15 to 15)

— real part
— imaginary part

Alternate form of the integral:

$$\frac{1}{2}\, x^2 \log(x) - \frac{x^2}{4} + \text{constant}$$